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LETTERS AND COMMENTS

Reply to Redžić's Comment: Electrostatic charges in $v \times B$ fields without special relativity

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Abstract
When a globally neutral conductor is moved in a static magnetic field, the charge carriers are transported by the electromotive field $\boldsymbol{v} \times \boldsymbol{B}$. The appearance of a density of free charge can be understood without invoking special relativity.

When an electric conductor immersed in a static magnetic field $\boldsymbol{B}$ is set into motion, an electric field is induced in the conductor. Lorrain remarked that, concomitantly, a free-charge density ρ is induced (Lorrain 1990, Lorrain *et al* 1998). Depending on the geometries of the magnetic field and the conductor, ρ may or may not be accompanied by an electric current density $\boldsymbol{J}$, the so-called induced current. I gave a non-relativistic account of the phenomenon (Bringuier 2003), wherein the electric field $\boldsymbol{E}'$ in the reference frame of the solid conductor is the sum of the electromotive field $\boldsymbol{v} \times \boldsymbol{B}$ ($\boldsymbol{v}$ is the conductor's velocity field) and of the 'electrostatic' field $\boldsymbol{E}$ created by the induced charge ρ (quotation marks are used because ρ and $\boldsymbol{E}$ are co-moving with the solid, rather than static). Lorrain worked out, *inter alia*, the example of a conducting sphere rotating around a diameter parallel to a uniform $\boldsymbol{B}$, and found that $\boldsymbol{E}$ cancels $\boldsymbol{v} \times \boldsymbol{B}$, making $\boldsymbol{E}'$ and $\boldsymbol{J} = \sigma \boldsymbol{E}'$ vanish (σ is the electrical conductivity). I worked the example of a circular conducting loop of thin wire, rotating around a diameter perpendicular to a uniform $\boldsymbol{B}$. I found that $\boldsymbol{E}$ cancels the radial component $(\boldsymbol{v} \times \boldsymbol{B})_\perp$ of $\boldsymbol{v} \times \boldsymbol{B}$, so that $\boldsymbol{E}'$ is directed along the wire, just like $\boldsymbol{J}$.

In his comment, Redžić (2004) notes that, in the latter example, my calculation of $\boldsymbol{E}$ from the surface charge density Σ of the wire is missing a factor of two. I agree with him. He also criticizes my calculation of Σ from $\boldsymbol{v} \times \boldsymbol{B}$, as the field $\boldsymbol{v} \times \boldsymbol{B}$ in vacuum is undefined, rather than vanishing. To prove my conclusion that $\boldsymbol{E}_\perp$ cancels $(\boldsymbol{v} \times \boldsymbol{B})_\perp$ is true, I will use a different line of reasoning. Firstly, the responses $\boldsymbol{E}$ and $\boldsymbol{J}$ have the same symmetries as the excitation $\boldsymbol{v} \times \boldsymbol{B} = -\boldsymbol{\omega}(\boldsymbol{B} \cdot \boldsymbol{r})$, where $\boldsymbol{\omega}$ is the angular velocity and $\boldsymbol{r}$ the position relative to the loop's centre. Symmetry with respect to the loop's plane entails that $\boldsymbol{E}_\perp$ and $\boldsymbol{J}_\perp$ are directed along $\boldsymbol{r}$. Secondly, we are dealing with a steady-state situation where div $\boldsymbol{J} = 0$. Since $\boldsymbol{J} = 0$ in vacuum outside the wire, the bulk equation div $\boldsymbol{J} = 0$ entails the surface equation $J_\perp = 0$ (no current crosses the surface of the wire). As the material is taken to be ohmic,

$$0 = \boldsymbol{J}_\perp = \boldsymbol{\sigma}(\boldsymbol{E}_\perp + (\boldsymbol{v} \times \boldsymbol{B})_\perp)$$

shows that the 'electrostatic' field cancels the radial component of the electromotive field. QED. As for the surface density Σ of the 'electrostatic' charges, it is obtained from the surface equation derived from the bulk one $\rho = \varepsilon_0 \operatorname{div} \boldsymbol{E}$, given that polarization (bound) charges play no role as $\operatorname{div} \boldsymbol{P} = (\varepsilon - \varepsilon_0) \operatorname{div}(\boldsymbol{J}/\sigma) = 0$ in an ohmic medium at steady state:

$$\Sigma = -\varepsilon_0 \boldsymbol{n} \cdot \boldsymbol{E}_\perp,$$

where $\boldsymbol{n}$ is the local unit vector pointing outwards from the wire. Thus, although setting $\boldsymbol{v} \times \boldsymbol{B}$ equal to zero in vacuum was somewhat open to criticism, my former equation (16) happens to be true. The actual flaw in my earlier calculation stems from the fact that, while the toric wire may be considered as locally cylindrical in the limit $R \gg \varnothing$ (R is the loop's radius and $\varnothing$ the wire's diameter), Σ varies with position and distant surface charges make a significant contribution to $\boldsymbol{E}$. *The uniform-Σ result is not applicable*[1].

Here are replies to other remarks raised in the Comment.

(i) $\boldsymbol{B}$ does not have the same physical meaning in equations (1) and (3) in the Comment. In the former, $\boldsymbol{B}$ is an external field imposed on the moving medium and giving rise to the electromotive field $\boldsymbol{v} \times \boldsymbol{B}$. In the latter, $\boldsymbol{B}$ is created by the conduction-plus-convection current of free charges in the medium and by the current of bound charges $\textbf{curl}\, \boldsymbol{M}$. The quantities denoted by $\boldsymbol{B}$ in (1) and (3) are physically distinct because they have different sources.

(ii) It is well known that the conduction current $\boldsymbol{J}$ induced by the motion causes a magnetic field $\boldsymbol{B}_i$ giving rise to the *self*-induction phenomenon which I have explicitly discarded in my paper. Is it indeed 'a rather rough approximation'? In the limit $R \gg \varnothing$, the self-inductance of a circular loop is $L = \mu_0 R[\ln(16R/\varnothing) - 7/4]$ (Jackson 1999). The $\boldsymbol{B}_i$-flux $\phi_i = L(-\mathrm{d}\phi'/\mathrm{d}t)\sigma s/2\pi R$ should be compared with the $\boldsymbol{B}$-flux $\phi' = \pi R^2 B \sin\theta (s = (1/4)\pi\varnothing^2$ is the wire's cross section and $\mathrm{d}\theta/\mathrm{d}t = \omega)$. One obtains $|\phi_i/\phi'| \approx \mu_0 \omega\sigma\varnothing^2[\ln(16R/\varnothing) - 7/4]/8$. Taking Cu ($\sigma = 5.6 \times 10^7\ \Omega^{-1}\ \mathrm{m}^{-1}$), $R = 0.1$ m, $\varnothing = 10^{-3}$ m and $\omega/2\pi = 100$ Hz, we get $|\phi_i/\phi'| \approx 3 \times 10^{-2}$. It is seen that self-induction matters at high frequencies for which the conductor's skin depth approaches $\varnothing$.

(iii) Redžić asserts that 'even a uniformly rotating ring does not give rise to a stationary situation'. Yet electromechanical generators and motors attest to the contrary. Theoretically speaking, the time needed to reach the electrical stationary state is the dielectric relaxation time ε/σ, regardless of the problem's symmetries. That time is very short, even in poor conductors (Bringuier 2003, 2004). Unless inordinately high frequencies are considered, most practical situations involve quasi-stationary states where $\partial\rho/\partial t$ is negligible but ρ is not.

(iv) The equation of motion of a conduction electron inside the conductor used by Redžić (2002) is a relativistic generalization of Newton's second law *in vacuum*. It is incorrect for two reasons. Firstly, between two collisions a conduction electron does not travel with its free mass, but has an energy–pseudomomentum dispersion relation which only in simple cases (not in Cu) is isotropic and parabolic and replaceable by the notion of an effective mass m^*. Secondly, the collisions with the medium bring about an exchange of pseudomomentum which is often pictured as a friction force. Collisions are responsible for Ohm's law which follows from the equation of motion $\boldsymbol{v}_\mathrm{d} = \mu\boldsymbol{F}$ instead of $\mathrm{d}\boldsymbol{v}_1/\mathrm{d}t = \boldsymbol{F}/m^*$ ($\boldsymbol{v}_\mathrm{d}$ is the drift velocity, i.e. the instantaneous velocity $\boldsymbol{v}_1$ averaged over several collision times, μ the mechanical mobility, and $\boldsymbol{F}$ the applied force; both equations of motion are written in the reference frame of the conductor). The former equation entails a constant $\boldsymbol{v}_\mathrm{d}$ under constant $\boldsymbol{F}$, while the latter entails a uniformly accelerating $\boldsymbol{v}_1$. In a solid or a not-too-rarefied gas, the pertinent equation including the effect of the collisions is the former one, not the latter as used by Redžić.

Nowhere in my treatment of induction is special relativity invoked, so that discrepancies of second order in v are possible. One has been mentioned in my previous paper, namely the

[1] A similar correction is due in the case of a *dielectric* wire in section 6 of my paper. The field $\boldsymbol{E}$ in the dielectric wire is obtained from $\boldsymbol{E}$ in the conducting wire upon multiplication by $(\varepsilon/\varepsilon_0 - 1)/(\varepsilon/\varepsilon_0 + 1)$.

convection current $\rho \boldsymbol{v}$. Equation (4) of Redžić's Comment is another one, as $\mu_0 \boldsymbol{M}$ is easily shown to be of the order of $(v/c)^2 B$ owing to $|\boldsymbol{E}| \approx |\boldsymbol{v} \times \boldsymbol{B}|$. Clearly my treatment is not intended to deal with such relativistic corrections.

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